Darcy's Law

Laboratory 2

HWR 531/431
Darcy's Law

Introduction

In 1856, Henry Darcy, a French hydraulic engineer, published a report in which he described a series of experiments he had performed in an attempt to quantify the amount of water that would flow through a column of sand. The experimental apparatus used was similar to the one displayed below.

![Experimental apparatus for the illustration of Darcy's Law. (Adapted from Freeze, R.A. and Cherry, J.A., 1979)](image)

This was the first systematic study of the movement of water through a porous medium. In this setup, water is introduced to a soil column at a constant rate $Q$. Water exits the column at the same rate $Q$, creating a steady-state flow regime. Fluid pressure is measured at two points on the column, separated by a column distance $\Delta l$ and a vertical distance $(z_1-z_2)$. The results of Darcy's experiments indicated that the rate at which a fluid moves through a porous medium ($Q$) is proportional to the difference in hydraulic head of the water along the column and the characteristics of the porous medium and the column length.

This relationship is known as **Darcy's law**:

$$Q = -KA \frac{dh}{dl}$$

the parameters of this equation are listed in Table 1 below:
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Dim</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discharge</td>
<td>Q</td>
<td>[L³/T]</td>
<td>the amount of fluid flowing past a point per unit time</td>
</tr>
<tr>
<td>Saturated Hydraulic</td>
<td>K</td>
<td>[L/T]</td>
<td>a coefficient of proportionality describing the rate at which a fluid can move through a permeable medium, dependant upon the density and viscosity of the fluid as well as the intrinsic permeability of the aquifer material.</td>
</tr>
<tr>
<td>Cross-section Area</td>
<td>A</td>
<td>[L²]</td>
<td>the area through which a volume of fluid passes</td>
</tr>
<tr>
<td>hydraulic head</td>
<td>h</td>
<td>[L]</td>
<td>the energy potential of the fluid at any point along the column</td>
</tr>
<tr>
<td>length</td>
<td>l</td>
<td>[L]</td>
<td>the distance (in a single direction) over which the measured change in hydraulic head is measured</td>
</tr>
<tr>
<td>hydraulic gradient</td>
<td>( \frac{dh}{dl} )</td>
<td>[-]</td>
<td>the change in total hydraulic head (h) over a given distance (l) in a given direction</td>
</tr>
</tbody>
</table>

Table 1: Description of Darcy's law parameters

During the Darcy experiment, constant head is maintained at either end of the column containing the porous medium. The discharge through the column is proportional to the change in head along the column and to the cross sectional area and inversely proportional to the length of the column.

\[ Q \propto A \frac{\Delta h}{\Delta l} \]

The constant of proportionality for this equation is called the hydraulic conductivity (K)

\[ Q = -KA \frac{dh}{dl} \]

Darcy's Law in one dimension

It is more convenient to write the discharge in terms of a flux (q) called the *specific discharge*:

\[ q = \frac{Q}{A} \]

with dimensions of velocity [L/T]

Darcy's Law becomes:

\[ q = -K \frac{dh}{dl} \]
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Although the specific discharge ($q$) has the dimensions of velocity, technically it is not. $q$ is the volume of water passing through a unit cross sectional area of porous medium per unit time. Because flow occurs through a porous medium, fluid moves only through a portion of the cross sectional area. As we learned in last week's lab, this portion of the cross sectional area is described by the porosity ($n$). Therefore, the actual fluid velocity or seepage velocity ($v$) is calculated as:

$$v = \frac{q}{n} = -\frac{K}{n} \frac{dh}{dl}$$

Assumptions of Darcy's Law:

The form of Darcy's Law shown above is a simplified mathematical relationship that depends upon a number of assumptions. Whenever such assumptions are made in order to simplify a system so that it may be characterized mathematically, these assumptions must be clearly stated and thoroughly explained (particularly in a laboratory report).

(i) **Steady flow:** It is assumed that flow is at a steady-state or equilibrium condition. The above form of Darcy's law assumes that the specific discharge is in steady-state and not time-dependent.

(ii) **Constant temperature:** Because hydraulic conductivity is related to the density and viscosity of the fluid, the temperature must be constant in order for Darcy's law to be valid. If temperature changes with time, conductivity is also a function of time. Because groundwater temperatures do not show large variability this assumption is valid for most groundwater systems.

(iii) **Laminar flow:** Darcy's law requires that flow be laminar because it does not contain terms to account for the non-linear dissipation of energy through turbulent flow.

(iv) **Uniform media/ single fluid:** In order for this relation to hold, the hydraulic conductivity must be constant for the media. Because the conductivity depends on the permeability of the...
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medium, non-uniform media will have variable hydraulic conductivity. Because it also depends upon the properties of the fluid, only single phase flow is valid

(v) *Incompressible fluid:* Darcy's law holds only if hydraulic head varies linearly with changes in elevation. For fluids that are easily compressible, this assumption is not valid

(vi) *Constant cross sectional area:* A variable cross sectional area would make the gradient variable

**Range of Validity of Darcy's Law**

Darcy's Law is only valid under the range of conditions where the above assumptions are valid. This means that the relation is valid for a variety of conditions and environments. However, it is important to recognize the limitations of Darcy's Law and the conditions under which we must be careful to examine our assumptions and the theory behind the law.

Because slow moving fluids are dominated by viscous forces, they have a low energy level and their flow is laminar. During laminar flow, molecules of water follow smooth lines called streamlines. (Figure 2A)

![Figure 2A: Laminar flow with parallel streamlines](image)

If the fluid velocity is increased it gains kinetic energy and flow becomes unstable. At some point, inertial forces dominate over the viscous forces and turbulent flow results. During turbulent flow, molecules of water no longer move along parallel streamlines. (Figure 2B)
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Figure 2B: Turbulent flow without parallel streamlines

The balance between viscous and inertial forces is expressed by a non-dimensional parameter called the

Reynolds number.

\[ R = \frac{\rho v d}{\mu} \]

R = Reynolds number, \( \rho \) = fluid density, \( v \) = fluid velocity, \( d \) = the diameter of the passageway through which the fluid moves, \( \mu \) = the dynamic viscosity of the fluid

At high Reynolds numbers, the relationship between specific discharge and hydraulic gradient is no longer linear. Therefore, the transition zone of Reynolds numbers in the range 1 to 10 is associated with the upper limit of Darcy's law. Such circumstances may occur in fracture flow and adjacent to wells that are pumping at high rates. Figure 3 illustrates the departure of flow from linearity that occurs at high Reynolds numbers.

Figure 3: Range of validity of Darcy's law (from Freeze and Cherry, 1979)
The lower limit of Darcy's law is associated with extremely slow groundwater movement. In such cases, other gradients, such as thermal, chemical and/or electrical, may be stronger than the hydraulic gradient and may control the movement of flow.

**Hydraulic Conductivity ($K$)**

Physical meaning:

As mentioned previously, hydraulic conductivity is "a coefficient of proportionality describing the rate at which water can move through a permeable medium. The density and kinematic viscosity of the water must be considered in determining hydraulic conductivity." Conductivity is thus not only a function of the properties of the porous medium but also of the fluid. However, in most natural groundwater systems, the fluid is water. Although the density and viscosity of water are temperature dependent, we can assume that the water remains at a constant temperature in most groundwater systems. Thus, hydraulic conductivity is usually reported as a property of the aquifer material.

The formula for hydraulic conductivity may be written as follows:

$$K = C \frac{d^2 \rho g}{\mu}$$

or

$$K = \frac{k \rho g}{\mu}$$

where:

- $C$ = proportionality constant dependent on grain size distribution, surface roughness, grain shape and packing
- $d$ = mean grain diameter ($d^2$ is related to the surface area in contact with the fluid)
- $\mu$ = viscosity of the fluid
- $\rho$ = density of the fluid
- $g$ = gravitational acceleration
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\[ k = \text{intrinsic permeability} = Cd^2 \]  
this parameter depends only on the properties of the porous medium and should be used with fluids that have properties that differ significantly from those of water.

**Navier Stoke's approximation:**

The system of flow through a porous medium can be approximated by a system of capillary tubes. In this way we can derive the above equations for Darcy's law and hydraulic conductivity. If we have a collection of \( m \) tubes of radius \( a \) in a box of cross sectional area \( B \), as shown below (Figure 5), we can calculate the area of these tubes by:

\[ A_{\text{tubes}} = m\pi a^2 \]

the porosity of this collection of tubes is:

\[ n = \frac{A_{\text{tubes}}}{A_{\text{total}}} = \frac{m\pi a^2}{B} \]

the number of tubes is thus:

\[ m = \frac{nB}{\pi a^2} \]

the total flux through the tubes is thus:

\[ Q_T = mQ = \frac{m\pi a^2}{B}Q \]

\[ Q_T = -B\frac{na^2}{8\mu} \frac{\partial P}{\partial x} \]

Note that this expression is similar to Darcy's Law.
There is a wide variability in values for hydraulic conductivity of water in different geological materials.

Figure 4 gives an approximate range of values for the hydraulic conductivity of different materials.

![Figure 4: Range of values of Hydraulic conductivity and permeability for various aquifer materials](image)

**Flow in layered systems**

Although Darcy's Law assumes a uniform medium, the real world is not so perfect. All natural systems are anisotropic and heterogeneous. Figure 5 displays the difference between these two aquifer characteristics.
Fortunately, in many geological systems, a portion of the anisotropy in hydraulic conductivity can be attributed to the presence of depositional layers of varying lithology (layered heterogeneity). In such cases, flow may be described as either parallel to the layers (parallel flow) or perpendicular to the layers (normal flow).

Flow parallel to layering:

The effective hydraulic conductivity for flow parallel to layering is calculated as a weighted arithmetic mean.
Darcy’s Law

\[ K = \frac{\sum K_i b_i}{\sum b_i} \]

Flow normal to layering:

By the law of conservation of mass, discharge through each layer must be the same and the system is dominated by the least permeable layer.

![Diagram of Series flow](image)

Figure 7: Illustration of Series flow. Because flow is across the layers, the effective K is a depth-weighted harmonic mean.

The effective hydraulic conductivity is calculated as a weighted harmonic mean.

\[ K = \frac{\sum b_i}{\sum b_i / k_i} \]
Darcy's Law

Hydraulic gradient

The hydraulic head gradient is the driving force for flow in most groundwater systems. It is related to a difference in potential energy. We can regard the hydraulic head at a point as the amount of potential energy contained in that parcel of fluid. Since energy potential is the work done per unit mass of fluid in transferring that mass of fluid from a reference state to a new state, we can derive an expression for hydraulic head based on first principles.

From Newton's First Law we have:

\[ F = ma \]

We also know that:

\[ W = Fd \]

And from above, we stated that:

\[ \Phi = \frac{W}{m} \]

To calculate the work done on a mass m, we can look at the three types of work acting on a fluid:

(i) Work against gravity

\[ W_g = mg(z - z_0) \]

(ii) Work to accelerate the fluid

\[ W_v = \frac{m(v - v_0)^2}{2} \]

(iii) Work to change the pressure from \( P_0 \) to \( P \)

\[ W_p = m \int_{P_0}^{P} \frac{1}{\rho} dP \]

Total Work is the sum of these three terms.
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\[ W = mg(z - z_0) + \frac{m(v - v_0)^2}{2} + m\int_{r_0}^{r} \frac{1}{\rho} dP \]

Energy potential is work/mass

\[ \Phi = \frac{W}{m} = g(z - z_0) + \frac{(v - v_0)^2}{2} + \int_{r_0}^{r} \frac{1}{\rho} dP \]

This is the Bernoulli Equation

Simplifications of the Bernoulli Equation:

In groundwater systems we can simplify this equation using two assumptions:

(i) Groundwater velocities are small enough such that velocity potential is negligible relative to other terms

\[ (v - v_0) = 0 \]

(ii) Fluid compressibility is small such that changes in density due to changes in pressure may be neglected

\[ \int_{r_0}^{r} \frac{1}{\rho} dP = \frac{1}{\rho} \int_{r_0}^{r} dP \]

Thus we can write an equation for the hydraulic head where hydraulic head expresses the work per unit weight of a fluid.(divide everything by g)

\[ h = \frac{\Phi}{g} = (z - z_0) + \frac{P - P_0}{\rho g} \]

\[ h = z + \Psi \]

\[ h = \text{elevation head} + \text{pressure head} \]

The gradient in hydraulic head is just the amount head changes over the path the fluid follows.
Laboratory Assignment

Introduction:

For most practical purposes, hydraulic conductivity of soils is best measured in-situ via a pump test or similar field method. However, sometimes laboratory tests of hydraulic conductivity are more desirable. In such cases, it is important to recognize the importance of soil structure and packing. Changes in these soil properties can cause large changes in the measured hydraulic conductivity. A number of methods may be used to collect soil cores intact and use them in soil column experiments. For a thorough discussion of procedures for obtaining undisturbed samples see McIntyre (1974).

The scale of the experiment is another important parameter which should be examined carefully. Generally, the dimensions of the samples should be comparable to the size of the largest geological units under investigation.

In some cases, repacked soil columns may be sufficient or even more useful than undisturbed soil columns. For example, repacked columns are more useful when studying the principles of flow in porous media. In such cases, uniformity in column packing is extremely important. For instructions on column packing, see Klute and Dirksen (1986).

Finally, it is critical that the entire sample be saturated because any trapped air bubbles will decrease the cross sectional area through which the water may pass, resulting in decreased conductivities. Also, the soil must be packed tightly so preferential pathways are not present along the sides of the column. This could result in increased conductivity values.

Experiment

In this laboratory we will be conducting permeameter tests to determine the hydraulic conductivity of two soils individually and the average conductivity of series flow through the two soils. The purpose of
this laboratory is to investigate the hydraulic properties of two repacked soils and to investigate the validity of the theories discussed in the reference section of the laboratory.

We will be using two laboratory methods to make our measurements. They are the falling-head permeameter and the constant-head permeameter. There will be five separate column experiments which each student will have to run. They are:

(1) Constant head test of soil A
(2) Constant head test of soil B
(3) Falling head test of soil A
(4) Falling head test of soil B
(5) Constant head test of layered column A-B

Below are guidelines for running the two types of permeameters. Make sure that you record all the information that you need.
Constant Head Permeameter

This experiment applies Darcy's law to a saturated soil column of uniform cross-sectional area to determine the hydraulic conductivity. In this case, a constant hydraulic gradient is applied to the soil column and the resulting flux of water is measured. Using this method, the hydraulic conductivity is given by the formula:

\[ K_s = \frac{Q L}{A (H_2 - H_1)} \]

where \( Q \) = volume water/time, \( L \) = length of column, \( A \) = cross sectional area, \((H_2 - H_1)\) = the change in head over the column length.

In Figure 7, a constant head of water is applied to a soil column using an overflow tank. Water moves through the column at a steady rate. The difference in head across the soil column is equal to the length \( H \). This difference in head is applied over a length of soil \( L \). The flow rate of water into and out of the column is \( Q \). The cross sectional area of the soil column is \( A \).

Figure 7: A constant head permeameter

Constant head permeameters are used for noncohesive soils such as sand and gravel. It is important to maintain a gradient similar to that observed in the field. A good measuring stick is that the head difference...
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should never be more than 50% of the sample length. The reason for this is that large gradients can produce high flow velocity which may result in high Reynolds numbers, turbulent flow and invalidation of Darcy's Law.

Laboratory Procedure:

(1) Examine the sand column. Look for any layers of different sized particles, bridging or any other indications of heterogeneity. Note the average grain size.

(2) Measure the length and diameter of the column

(3) Record the elevation of the input constant head flask above an arbitrary datum.

(4) Record the elevation of the output tube above the arbitrary datum.

(5) Establish a steady flow through the column.

(6) Measure the flow rate using a graduated cylinder and a stopwatch (*make multiple measurements*)
Falling head permeameter

The falling head permeameter method is used for soils that are more cohesive and with lower hydraulic conductivity. In this experiment, much smaller volumes of water flow through the sample.

![Falling head permeameter diagram](image)

**Figure 8: A falling head permeameter**

In the falling head permeameter, the hydraulic head over the soil column is allowed to decrease a distance $H_0 - H_1$ in time $t$. The cross sectional area of the falling head tube and the cross sectional area of the soil column are known (and noted if different). The conductivity is thus found using the formula:

$$K = \frac{d_t^2 L}{d_i^2 t} \ln \left( \frac{h_0}{h} \right)$$

where $d_c = \text{diameter of the soil column}$, $d_t = \text{diameter of the falling head tube}$, $h_0 = \text{initial head}$, $h = \text{final head}$, $L = \text{length of soil column}$, $t = \text{time to drop distance } h_0-h$. 

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Laboratory procedure

(1) Examine the sand column. Look for any layers of different sized particles, bridging or any other indications of heterogeneity. Note the average grain size.

(2) Measure the length and diameter of the column and if necessary, the diameter of the falling head tube.

(3) Record the elevation of the output constant head flask above an arbitrary datum

(4) Clamp off the hose leading to the output constant head flask.

(5) Initially, the water level will be approximately 2cm above the surface of the sand. Make sure that the water level is always above the top of the sand. Bring the water level 35-40 cm above the top of the sand, fill the constant head flask and position it in the sink.

(6) Tap the column so that the upper sand surface is level

(7) Note the initial height of the water in the column

(8) Release the outlet hose clamp and measure the position of the water surface every 5 seconds until the water level is back to the initial level. Repeat the test if the result are unsatisfactory.
Lab Report:

The following should be included in the laboratory report. The format should be that of a scientific paper. It should be typed with all appropriate tables, graphs and figures produced on computer.

(1) Abstract

(2) Background: discuss the principles of Darcy's law and the use of permeameters in determining hydraulic conductivity. Derive the equations for falling head and constant head permeameters.

(4) Theory and procedures: discuss what was done in the lab and why.

(4) Discussion: take a critical look at the data, using the guidelines below

(5) Conclusions: What did you learn? What recommendations can you make about testing hydraulic conductivity in the lab. (Not recommendations for our lab but recommendations in general)

(6) Include diagrams and charts, explain the experimental setup. If the setup is different than the one in the laboratory report, explain how it is different and compare the two. Make tables of the recorded data that was recorded

(5) Include a separate page of calculations showing your work
Guidelines:

(1) Calculate hydraulic conductivity using the formulas for each type of permeameter.

(2) Calculate hydraulic conductivity from the slope of the line from a plot of gradient versus discharge. If the relation is not linear, choose the points closest to the origin.

(3) Calculate seepage velocities and Reynolds numbers.

(4) Compare values, discuss Darcy's law assumptions/validity.

(5) Calculate average conductivity for normal flow and compare to measured values.

(6) Discuss all these results thoroughly and tabulate the values calculated.